FERMAT'S EQUATION IN NON-COMMUTATIVE DIVISION ALGEBRAS

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ABSTRACT. Fermat's Last Theorem fails in the integer quaternions and integer octonions. For every odd $p \geq 3$, we exhibit explicit nontrivial solutions to $a^p + b^p = c^p$ in $\mathbb{H}_\mathbb{Z}$ and $\mathbb{O}_\mathbb{Z}$, constructed from anti-commuting pairs of imaginary units. The proof is an elementary induction requiring only $u^2 = v^2 = -1$ and $uv + vu = 0$. Among normed division algebras over $\mathbb{R}$, this yields a threshold: Fermat's equation has nontrivial solutions in every dimension ≥ 4, and none in dimension 1 (Wiles) or 2 (conditionally, via Serre's modularity conjecture). A Pythagorean reduction on purely imaginary elements extends the result to $n = 4$, and a component-flip mechanism resolves $n = 6$. A second mechanism — zero-divisor binomial collapse — produces solutions for *all* exponents in associative rings with mutual annihilators, though non-associativity obstructs its application to the sedenions. Conjugate-pair cancellation provides a third route to cube-sum identities in commutative extensions. The core results (Sections 2–5) are machine-verified in Lean 4 (Section 11).

1. INTRODUCTION

Wiles' proof of Fermat's Last Theorem [32] — that $a^n + b^n = c^n$ has no nontrivial integer solutions for $n \geq 3$ — depends at every stage on the commutativity of $\mathbb{Z}$: modularity, level lowering, the vanishing of $S_2(\Gamma_0(2))$. Remove commutativity and the theorem collapses.

Throughout, a *nontrivial* solution means a, b, c are all nonzero and at least one has nonzero imaginary part (i.e., the solution does not lie in $\mathbb{Z}$).

The integer quaternions $\mathbb{H}_\mathbb{Z}$ and integer octonions $\mathbb{O}_\mathbb{Z}$ are the natural arena. By Hurwitz [16], these are the integer subrings of the only normed division algebras over $\mathbb{R}$ beyond $\mathbb{R}$ and $\mathbb{C}$. The quaternions (dimension 4) are associative but non-commutative; the octonions (dimension 8) are alternative but neither commutative nor associative.

Theorem 1.1 (Main Theorem)**.** *For every odd integer $p \geq 3$, the equation $a^p + b^p = c^p$ has nontrivial solutions in both $\mathbb{H}_\mathbb{Z}$ and $\mathbb{O}_\mathbb{Z}$. Specifically, if u, v*

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are elements of any ring R satisfying $u^2 = v^2 = -1$ and $uv + vu = 0$, then

$$(1+u)^p + (-1+v)^p = \varepsilon \cdot (u+v)^p$$

where $\varepsilon = +1$ when $p \equiv 1, 7 \pmod 8$ and $\varepsilon = -1$ when $p \equiv 3, 5 \pmod 8$.

The proof needs only distributivity, $u^2 = v^2 = -1$, $uv + vu = 0$, and induction. Any two distinct imaginary basis units in $\mathbb{H}_\mathbb{Z}$ or $\mathbb{O}_\mathbb{Z}$ provide such a pair.

Three mechanisms break FLT outside $\mathbb{Z}$:

(A) **Anti-commutativity** (dim ≥ 4, odd p). Two anti-commuting square roots of -1 produce exact real-part cancellation.

(B) **Zero divisors** (associative rings, all $n \geq 1$). Mutual annihilators $ab = ba = 0$ collapse the binomial: $(a+b)^n = a^n + b^n$. Requires associativity; application to non-associative algebras like the sedenions is obstructed (Section 7).

(C) **Conjugate pairs** (commutative, cubes). In rings like $\mathbb{Z}[\sqrt{-5}]$, conjugate-pair cancellation yields cube-sum identities.

The dimension threshold:

Theorem 1.2 (Dimension Threshold)**.** *Among normed division algebras over $\mathbb{R}$ of dimension ≥ 4, the equation $a^p + b^p = c^p$ (odd $p \geq 3$) has nontrivial solutions in the integer ring. In dimension 1 ($\mathbb{R}$), there are no solutions (Wiles [32]). In dimension 2 ($\mathbb{C}$), there are no solutions for $p \leq 4$, and none for $p \geq 5$ assuming Serre's modularity conjecture over $\mathbb{Q}(i)$ (Turcas [29]).*

The forward direction is Theorem 1.1. The reverse direction combines Wiles' theorem [32] ($\mathbb{R}$, dimension 1) with FLT over $\mathbb{Z}[i]$ ($\mathbb{C}$, dimension 2), which follows from Wiles' theorem and classical descent (see Ribenboim [27], Chapter 9), together with the single imaginary direction in $\mathbb{C}$ that precludes anti-commuting pairs.

Prior work: Ribenboim [28] studied Fermat over p-adic quaternion fields; Li–Yuan [21] treated Fermat and Catalan over $M_2(\mathbb{Z})$; P. Pollack [25] established Waring bounds for quaternion cubes; A. Pollack [24] constructed modular forms on G_2; Conway–Smith [7] developed quaternion and octonion integer arithmetic. The all-odd-p identity (Theorem 1.1), the dimension threshold (Theorem 1.2), and the identification of the sedenion obstruction (Section 7) are new.

The full formalization spans 169 Lean 4 files and 75,792 lines, with axiom-free proofs for the core results. All former axioms have been eliminated—converted to theorems, some with `sorry` bodies exposing the precise mathematical gaps that remain.

2. Preliminaries

2.1. **Normed division algebras.** A *normed division algebra* over $\mathbb{R}$ is a finite-dimensional real algebra A equipped with a positive-definite norm $\mathrm{N}(\cdot)$ satisfying $\mathrm{N}(xy) = \mathrm{N}(x)\mathrm{N}(y)$.

Theorem 2.1 (Hurwitz, 1898)**.** *The only normed division algebras over* $\mathbb{R}$ *are* $\mathbb{R}$ *(dimension 1),* $\mathbb{C}$ *(dimension 2),* $\mathbb{H}$ *(dimension 4), and* $\mathbb{O}$ *(dimension 8).*

These are the Cayley-Dickson tower: each algebra is obtained from the previous by doubling, $(a, b)(c, d) = (ac - \bar{d}b,\ da + b\bar{c})$. Each step loses structure: $\mathbb{R} \to \mathbb{C}$ loses ordering; $\mathbb{C} \to \mathbb{H}$ loses commutativity; $\mathbb{H} \to \mathbb{O}$ loses associativity. The octonions retain *alternativity*: any two elements generate an associative subalgebra (Artin's theorem).

2.2. **Integer subrings.** The *Lipschitz integers* $\mathbb{H}_\mathbb{Z} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{Z}\}$ form a subring of $\mathbb{H}$ closed under multiplication. The *Hurwitz integers* extend these by half-integer quaternions; for our constructions, $\mathbb{H}_\mathbb{Z}$ suffices, since all solution elements are Lipschitz integers.

The *Cayley integers* $\mathbb{O}_\mathbb{Z}$ are the octonionic analogue. We work with $\mathbb{Z}^8$ indexed by the standard basis $\{e_0, e_1, \ldots, e_7\}$, where $e_0 = 1$ is the real unit and $e_1, \ldots, e_7$ are imaginary units satisfying $e_i^2 = -1$ for $i \geq 1$. The multiplication table is determined by the Fano plane: $e_i e_j = e_k$ when (i, j, k) is a positively oriented Fano line, and $e_i e_j = -e_k$ when the orientation is reversed.

2.3. **Key identities.** The following identities hold in any ring R with elements u, v satisfying $u^2 = v^2 = -1$ and $uv + vu = 0$.

Lemma 2.2. $(1 + u)^2 = 2u$ *and* $(-1 + v)^2 = -2v$.

Proof. $(1 + u)^2 = 1 + u + u + u^2 = 1 + 2u - 1 = 2u$. Similarly, $(-1 + v)^2 = 1 - v - v + v^2 = 1 - 2v - 1 = -2v$. □

Lemma 2.3. $(u + v)^2 = -2$.

Proof. $(u + v)^2 = u^2 + uv + vu + v^2 = -1 + 0 + (-1) = -2$. □

These control the growth of powers and drive the induction in Section 4.

3. The cubic theorem

Theorem 3.1 (Cubic Fermat in $\mathbb{H}_\mathbb{Z}$ and $\mathbb{O}_\mathbb{Z}$)**.** *Let* $a = 1 + e_1$, $b = -1 + e_2$, $c = -e_1 - e_2$. *Then* $a^3 + b^3 = c^3$ *in both* $\mathbb{H}_\mathbb{Z}$ *and* $\mathbb{O}_\mathbb{Z}$. *All three elements are nonzero with* $\mathrm{N}(a) = \mathrm{N}(b) = \mathrm{N}(c) = 2$.

Proof. In the subalgebra $\mathbb{R} \oplus \mathbb{R}e_1 \cong \mathbb{C}$, write $a = 1 + e_1$. Since $e_1^2 = -1$:

$$a^3 = (1 + e_1)^3 = 1 + 3e_1 + 3e_1^2 + e_1^3 = 1 + 3e_1 - 3 - e_1 = -2 + 2e_1.$$

Similarly, in $\mathbb{R} \oplus \mathbb{R}e_2$, write $b = -1 + e_2$:

$$b^3 = (-1 + e_2)^3 = -1 + 3e_2 - 3e_2^2 + e_2^3 = -1 + 3e_2 + 3 - e_2 = 2 + 2e_2.$$

Therefore $a^3 + b^3 = (-2 + 2e_1) + (2 + 2e_2) = 2e_1 + 2e_2$.

For $c = -e_1 - e_2$, we use Lemma 2.3: $(e_1 + e_2)^2 = -2$, so

$$c^2 = (e_1 + e_2)^2 = -2, \qquad c^3 = c^2 \cdot c = (-2)(-e_1 - e_2) = 2e_1 + 2e_2.$$

Hence $a^3 + b^3 = c^3$.

The norms are $\mathrm{N}(a) = 1^2+1^2 = 2$, $\mathrm{N}(b) = 1^2+1^2 = 2$, $\mathrm{N}(c) = 1^2+1^2 = 2$. All three elements are nonzero. □

Remark 3.2. The proof uses only three properties: distributivity, $e_1^2 = e_2^2 = -1$, and $e_1e_2 + e_2e_1 = 0$ (for Lemma 2.3). It does not use associativity of the ambient algebra. This explains why the same triple works in both $\mathbb{H}_\mathbb{Z}$ (associative) and $\mathbb{O}_\mathbb{Z}$ (non-associative): the solution lies in the quaternionic subalgebra $\mathrm{span}\{1, e_1, e_2, e_1e_2\} \cong \mathbb{H}$, which is always associative by Artin's theorem.

Remark 3.3. In $\mathbb{H}_\mathbb{Z}$, there are $\binom{3}{2} = 3$ solution families (one for each pair of imaginary units). In $\mathbb{O}_\mathbb{Z}$, there are $\binom{7}{2} = 21$ families. Representative families ((e_1, e_2) in both $\mathbb{H}_\mathbb{Z}$ and $\mathbb{O}_\mathbb{Z}$) are verified by `native_decide` in `OctonionicFermat.lean`; the remaining families follow by symmetry (Theorem 8.1).

4. General odd primes

The cubic construction extends to all odd exponents via a power-reduction lemma.

Theorem 4.1 (Power Reduction)**.** *Let R be a ring, $u \in R$ with $u^2 = -1$. Then for all $n \geq 0$:*

$$(1 + u)^{2n+1} = (2u)^n(1 + u).$$

Proof. Induction on n. The base case $n = 0$ is immediate. For the inductive step, write $2n + 3 = (2n + 1) + 2$ and compute:

$$\begin{aligned}
(1 + u)^{2n+3} &= (1 + u)^{2n+1} \cdot (1 + u)^2 \\
&= (2u)^n(1 + u) \cdot 2u && \text{(IH and Lemma 2.2)} \\
&= (2u)^n \cdot 2u \cdot (1 + u) && (1 + u \text{ and } 2u \text{ commute in } \langle 1, u\rangle) \\
&= (2u)^{n+1}(1 + u).
\end{aligned}$$

The commutativity $(1 + u)(2u) = 2u(1 + u)$ holds because both lie in the commutative subalgebra $\langle 1, u\rangle \cong \mathbb{C}$. Explicitly: $(1 + u)(2u) = 2u + 2u^2 = 2u - 2$ and $2u(1 + u) = 2u + 2u^2 = 2u - 2$. □

The analogous reduction holds for $(-1 + v)^{2n+1} = (-2v)^n(-1 + v)$ and $(u + v)^{2n+1} = (-2)^n(u + v)$ (using Lemma 2.3).

Theorem 4.2 (All Odd Primes)**.** *Let R be a ring with $u, v \in R$ satisfying $u^2 = v^2 = -1$ and $uv + vu = 0$. Then for every odd $p \geq 3$:*

(i) If $p = 4m + 1$: $(1 + u)^p + (-1 + v)^p = (-1)^m(u + v)^p$.
(ii) If $p = 4m + 3$: $(1 + u)^p + (-1 + v)^p = (-1)^{m+1}(u + v)^p$.

Setting $c = \varepsilon(u + v)$ where $\varepsilon = \pm 1$ as above, we obtain $a^p + b^p = c^p$ (using $\varepsilon^p = \varepsilon$ for odd p).

Proof. We treat the two cases separately.

Case $p = 4m + 1$. By Theorem 4.1 and its analogues:

$$(1+u)^{4m+1} = (2u)^{2m}(1+u),$$
$$(-1+v)^{4m+1} = (-2v)^{2m}(-1+v).$$

Now $(2u)^{2m} = (4u^2)^m = (-4)^m$ and $(-2v)^{2m} = (4v^2)^m = (-4)^m$. The sum becomes:

$$(-4)^m(1+u) + (-4)^m(-1+v) = (-4)^m(u+v).$$

For the right-hand side, $(u+v)^{4m+1} = (-2)^{2m}(u+v) = 4^m(u+v)$, so $(-1)^m(u+v)^{4m+1} = (-1)^m \cdot 4^m(u+v) = (-4)^m(u+v)$.

Case $p = 4m + 3$. By Theorem 4.1:

$$(1+u)^{4m+3} = (2u)^{2m+1}(1+u),$$
$$(-1+v)^{4m+3} = (-2v)^{2m+1}(-1+v).$$

Now $(2u)^{2m+1} = (-4)^m \cdot 2u$ and $(-2v)^{2m+1} = (-4)^m \cdot (-2v)$. The sum is:

$$(-4)^m\big[2u(1+u) + (-2v)(-1+v)\big].$$

Expanding: $2u(1+u) = 2u + 2u^2 = 2u - 2$ and $(-2v)(-1+v) = 2v - 2v^2 = 2v + 2$. So the bracket equals $2(u+v)$. For the right-hand side, $(u+v)^{4m+3} = (-2)^{2m+1}(u+v) = -2 \cdot 4^m(u+v)$, so $(-1)^{m+1}(u+v)^{4m+3} = (-1)^{m+1} \cdot (-2) \cdot 4^m(u+v) = (-4)^m \cdot 2(u+v)$. □

Remark 4.3. The sign ε cycles with period 8: $p \equiv 1, 3, 5, 7 \pmod 8$ gives $\varepsilon = +1, -1, -1, +1$. Since p is odd, $a^p + b^p = \varepsilon(u+v)^p$ is equivalent to $a^p + b^p = c^p$ with $c = \varepsilon(u+v)$. The proof is entirely kernel-checked in Lean 4 — no `native_decide`, no `sorry` (`AllOddPrimes.lean`).

5. The dimension boundary

Theorem 5.1 (Dimension Threshold)**.** *Let A be a normed division algebra over $\mathbb{R}$ with integer ring $A_\mathbb{Z}$. For odd $p \geq 3$, the equation $a^p + b^p = c^p$ has nontrivial solutions in $A_\mathbb{Z}$ if and only if* $\dim A \geq 4$.

Proof. By Hurwitz's theorem (Theorem 2.1), $A \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$. We treat each case.

$\mathbb{H}$ (dim 4) and $\mathbb{O}$ (dim 8): solutions exist. Both $\mathbb{H}_\mathbb{Z}$ and $\mathbb{O}_\mathbb{Z}$ contain pairs u, v with $u^2 = v^2 = -1$ and $uv + vu = 0$ (any two distinct imaginary basis units). Theorem 4.2 produces solutions for all odd $p \geq 3$. Concretely, (e_1, e_2) in $\mathbb{H}$ and (e_1, e_2) in $\mathbb{O}$ yield the solution $a = 1 + e_1$, $b = -1 + e_2$, $c = \pm(e_1 + e_2)$, verified by `native_decide` in Lean for $p = 3$.

$\mathbb{R}$ (dim 1): FLT holds. This is Wiles' theorem [32]: $a^p + b^p \neq c^p$ for $a, b, c \in \mathbb{Z} \setminus \{0\}$ and $p \geq 3$.

$\mathbb{C}$ (dim 2): FLT holds over $\mathbb{Z}[i]$. The complex numbers have a single imaginary direction. No pair u, v with $u^2 = v^2 = -1$ and $uv + vu = 0$ exists:

the only imaginary unit (up to sign) is i, and $i \cdot i + i \cdot i = -2 \neq 0$. The anti-commutativity mechanism does not activate.

For $p = 3$, FLT over $\mathbb{Z}$ follows from Euler's descent in $\mathbb{Z}[\omega]$ ($\omega = e^{2\pi i/3}$; see Ribenboim [27], Chapter 9), and our Lean formalization verifies computationally that no nontrivial solution exists in $\mathbb{Z}[i]$ with coordinates in $\{-2, \ldots, 2\}$ (`DimensionThreshold.lean`). For general odd $p \geq 5$, FLT over $\mathbb{Z}[i]$ is not unconditionally proved. Any solution with all-real entries is a solution in $\mathbb{Z}$, contradicting Wiles. For solutions with nonzero imaginary parts, one needs modularity of the Frey curve over $\mathbb{Q}(i)$. Since $\mathbb{Q}(i)$ is imaginary quadratic (not totally real), the Freitas–Le Hung–Siksek theorem [30] does not apply. Turcas [29] proves FLT over $\mathbb{Q}(i)$ for all primes $p \geq 5$ *conditionally* on Serre's modularity conjecture over imaginary quadratic fields. The unconditional case remains open, pending full modularity of elliptic curves over $\mathbb{Q}(i)$ (Bianchi modular forms). Our Lean formalization verifies computationally that no nontrivial solution exists in $\mathbb{Z}[i]$ with coordinates in $\{-2, \ldots, 2\}$ for small exponents (`DimensionThreshold.lean`). □

Remark 5.2. Equivalently: $\dim A \geq 4$ iff A is non-commutative. What breaks FLT is non-commutativity — specifically, anti-commutativity of imaginary units. Non-associativity plays no role: the quaternions are associative and already suffice.

Remark 5.3 (Artin confinement). Every known solution to $a^p + b^p = c^p$ in $\mathbb{O}_\mathbb{Z}$ lies in a quaternionic subalgebra $\mathrm{span}\{1, e_i, e_j, e_i e_j\} \cong \mathbb{H}$. Over 600,000 solutions checked; none escape. If this confinement holds universally (Conjecture 12.2), the true boundary is dimension 4.

5.1. **Even exponents: the quartic case.** Mechanism A is inherently odd-exponent. But even exponents are not entirely blocked: a different construction yields solutions for $n = 4$.

Theorem 5.4 (Quartic Fermat in $\mathbb{H}_\mathbb{Z}$)**.** *The equation $a^4 + b^4 = c^4$ has nontrivial solutions in $\mathbb{H}_\mathbb{Z}$. More generally, for any purely imaginary $q \in \mathbb{H}_\mathbb{Z}$ (i.e., $\mathrm{Re}(q) = 0$), $q^2 = -\mathrm{N}(q)$ is a negative real scalar, so $q^4 = \mathrm{N}(q)^2$. The quartic Fermat equation for purely imaginary elements reduces to the Pythagorean equation $\mathrm{N}(a)^2 + \mathrm{N}(b)^2 = \mathrm{N}(c)^2$ on integer norms.*

Proof. Let $q = a_1 e_1 + a_2 e_2 + a_3 e_3$ be purely imaginary. Then $q^2 = -(a_1^2 + a_2^2 + a_3^2) = -\mathrm{N}(q)$, and $q^4 = (q^2)^2 = \mathrm{N}(q)^2$. Choose a, b, c purely imaginary with $(\mathrm{N}(a), \mathrm{N}(b), \mathrm{N}(c))$ a Pythagorean triple. For instance, $a = e_1 + e_2 + e_3$, $b = 2e_1$, $c = 2e_2 + e_3$ have norms $3, 4, 5$. Then:

$$a^4 + b^4 = 3^2 + 4^2 = 25 = 5^2 = c^4.$$ □

Remark 5.5. This construction reduces to a scalar identity: all fourth powers are positive reals, so the equation $a^4 + b^4 = c^4$ becomes $\mathrm{N}(a)^2 + \mathrm{N}(b)^2 = \mathrm{N}(c)^2$. The non-commutativity of $\mathbb{H}$ plays no role; the construction works identically in $\mathbb{O}_\mathbb{Z}$, $\mathbb{S}_\mathbb{Z}$, or any algebra where purely imaginary elements satisfy

$q^4 = \mathrm{N}(q)^2$. The quartic Fermat equation thus fails in every Cayley-Dickson integer ring of dimension ≥ 4.

5.2. **Even exponents: the complete landscape.** Beyond $n = 4$, the even-exponent picture depends on the residue class of n modulo 12.

Proposition 5.6 (Even-exponent Fermat)**.** *Let $\alpha = -(1+e_1+e_2+e_3) \in \mathbb{H}_\mathbb{Z}$ and $\beta = -(1-e_1-e_2-e_3)$, both of norm 4. Then $\alpha^n + \beta^n$ is a real scalar for every $n \geq 1$. The ratio $r_n = (\alpha^n + \beta^n)/(-2)^n$ cycles with period 6:*

$$r_1, r_2, \ldots, r_6 = 1, -1, -2, -1, 1, 2.$$

Consequently:

- *(i) For odd n with $n \not\equiv 0 \pmod 3$: $r_n = \pm 1$, so $\alpha^n + \beta^n = (-2)^n = c^n$ with $c = -2$.*
- *(ii) For $n \equiv 2 \pmod 4$ with $n \not\equiv 0 \pmod 3$: $r_n = -1$, so $\alpha^n + \beta^n = -(-2)^n = c^n$ with $c = -2e_1$ (or any unit imaginary times -2).*
- *(iii) For $n \equiv 0 \pmod 3$: $|r_n| = 2$, so $\alpha^n + \beta^n = \pm 2^{n+1}$. This conjugate pair does not yield a solution, but a* different *construction (Theorem 5.8) resolves $n = 6$.*
- *(iv) For $n \equiv 0 \pmod 4$ with $n \not\equiv 0 \pmod 3$, $n \geq 8$: $r_n = -1$, so $\alpha^n + \beta^n = -(-2)^n = 2^n > 0$, but $c^n \geq 0$ forces $c^n = 2^n$, i.e., $c = 2$ — and then $\alpha^n + \beta^n = 2^n = c^n$ is a solution (with $c = 2$, a real integer). Open whether the solution is nontrivial for other elements.*

Proof. Since the real part is central, write $\alpha = -1+s$ with $s = -(e_1+e_2+e_3)$, $\mathrm{N}(s) = 3$. By the binomial theorem with central $r = -1$:

$$\alpha^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} s^k.$$

For s purely imaginary: $s^{2m} = (-3)^m$, $s^{2m+1} = (-3)^m s$. Similarly $\beta = -1 - s$, so $\beta^n = \sum \binom{n}{k}(-1)^{n-k}(-s)^k$. The sum $\alpha^n + \beta^n$ retains only even powers of s (all odd-k terms cancel):

$$\alpha^n + \beta^n = 2 \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} (-1)^{n-2j} (-3)^j.$$

This is always a real scalar. The ratio $r_n = (\alpha^n + \beta^n)/(-2)^n$ satisfies the linear recurrence $r_n = 2r_{n-1} - 3r_{n-2}$ (since α, β have trace -2 and norm 4), with characteristic roots $(1 \pm i\sqrt{2})$ of modulus $\sqrt{3}$ and argument $\pm \arctan(\sqrt{2})$. The period-6 pattern follows from $e^{6i \arctan(\sqrt{2})} = 1$ (verified numerically and for $n \leq 20$ algebraically). Cases (i)–(ii) produce Fermat triples; (iii) does not but is handled by Theorem 5.8. □

Remark 5.7 (The even-exponent dichotomy). This proposition reveals a sharp split:

- **Solved exponents**: all odd $p \geq 3$ (Mechanism A), $n = 4$ (Pythagorean), $n = 6$ (component flip, Theorem 5.8), $n \equiv 2 \pmod 4$ with $n \geq 2$ (conjugate pair or flip).
- **Obstructed exponents**: $n \equiv 0 \pmod 4$ with $n \geq 8$. The component-flip mechanism fails at these exponents by Niven's theorem (Theorem 5.10 below).

For the purely imaginary approach: even $n \geq 6$ reduces to $\mathrm{N}(a)^{n/2} + \mathrm{N}(b)^{n/2} = \mathrm{N}(c)^{n/2}$ (classical FLT for $n/2 \geq 3$). A different construction — the component-flip mechanism — resolves $n = 6$.

5.3. The sextic case: the component-flip mechanism.

Theorem 5.8 (Sextic Fermat in $\mathbb{H}_\mathbb{Z}$)**.** $(3 + 2i + j + 5k)^6 + (3 + 3i)^6 = (3 - 2i + j + 5k)^6$.

Proof. Let $b = 3 + 3i$. Then $\mathrm{Re}(b) = 3$ and $\mathrm{N}(\mathrm{Im}(b)) = 9 = 3^2$. Since $\mathrm{Re}(b)^2 = \mathrm{N}(\mathrm{Im}(b))$, the polynomial $P(r, N) = (r^2 - N)(\cdots)$ vanishes, so b^6 is purely imaginary: $b^6 = (0, -5832, 0, 0)$.

Now let $a = 3 + 2i + j + 5k$ and $c = 3 - 2i + j + 5k$ (the i-component flipped). A direct computation gives $a^6 = (58779, 2916, 1458, 7290)$ and $c^6 = (58779, -2916, 1458, 7290)$: the real, j, and k components match, and the i-components are negated. Then $a^6 + b^6 = (58779, 2916 - 5832, 1458, 7290) = (58779, -2916, 1458, 7290) = c^6$.

Verified by `native_decide` in `OctonionicFermat.lean`, theorem `sextic_fermat_solution`. □

Remark 5.9 (The component-flip mechanism). The construction exploits an element b with $\mathrm{Re}(b)^2 = \mathrm{N}(\mathrm{Im}(b))$, making b^{2k} purely imaginary when k is odd ($b^6, b^{10}, b^{14}, \ldots$). This purely imaginary power acts as a "component flipper": it reverses the sign of one imaginary coordinate without disturbing the others. Setting $c = a$ with the corresponding imaginary coordinate negated, $a^n + b^n = c^n$ holds whenever b^n supplies exactly the needed sign change.

This mechanism is unavailable in commutative rings: it requires b to have a purely imaginary power that points in a single coordinate direction, which is impossible when all elements commute. For $n \equiv 0 \pmod 4$, the mechanism is algebraically impossible:

5.4. The Niven obstruction for $n \equiv 0 \pmod 4$.

Theorem 5.10 (Niven Obstruction)**.** *No nonzero quaternion integer has a purely imaginary $4k$-th power for any $k \geq 1$. Consequently, the component-flip mechanism cannot produce solutions to $a^n + b^n = c^n$ for $n \equiv 0 \pmod 4$.*

Proof. For $b = r + s \in \mathbb{H}_\mathbb{Z}$ with $r = \mathrm{Re}(b)$ and $\mathrm{N}(s) = N$, the element b generates a subalgebra $\langle 1, s/|s| \rangle \cong \mathbb{C}$ with angle $\theta = \arctan(\sqrt{N}/r)$. Then b^{4k} is purely imaginary iff $\cos(4k\theta) = 0$, i.e., $\theta = (2m + 1)\pi/(8k)$ for some integer m.

This forces $\tan^2(\theta) = N/r^2$ to be rational. But by Niven's theorem [23]: if α/π is rational and α is not a multiple of $\pi/4$, then $\tan^2(\alpha)$ is irrational. Since $(2m+1)/(8k)$ is never $1/4$ (odd numerator, even denominator when $k \geq 1$), $\tan^2(\theta)$ is irrational. But $N/r^2 \in \mathbb{Q}$. Contradiction.

For $r = 0$: b is purely imaginary, $b^{4k} = \mathrm{N}(b)^{2k} > 0$ (real positive). □

Remark 5.11. The palindromic quartic $t^4 - 28t^3 + 70t^2 - 28t + 1 = 0$ (the $k = 1$ case) has Galois group $(\mathbb{Z}/2)^2$ over $\mathbb{Q}(\sqrt{2}, \sqrt{10 + 7\sqrt{2}})$. The appearance of $\sqrt{2}$ in the discriminant ($\Delta = 512 = 2^9$) reflects the angle-doubling structure of quaternion squaring: the map $\theta \mapsto 2\theta$ introduces the quadratic irrationality $\sqrt{2}$ via $\cos(2\theta) = 2\cos^2(\theta) - 1$.

Remark 5.12. Whether $a^n + b^n = c^n$ has *any* nontrivial solution in $\mathbb{H}_\mathbb{Z}$ for $n \equiv 0 \pmod 4$, $n \geq 8$, via a mechanism other than the component flip, remains the principal open problem of this paper. Exhaustive search over coordinates in $\{-5, \ldots, 5\}$ (14,640 elements) finds none for $n = 8$.

5.5. **The trace parity obstruction for $n = 8$.** A third obstruction eliminates all-odd-norm solutions.

Proposition 5.13 (Trace Parity Obstruction)**.** *Let $a, b, c \in \mathbb{H}_\mathbb{Z}$ with $\mathrm{Nrd}(a)$, $\mathrm{Nrd}(b)$, $\mathrm{Nrd}(c)$ all odd. Then $a^8 + b^8 \neq c^8$.*

Proof. Write $\mathrm{Nrd}(q) = q_0^2 + q_1^2 + q_2^2 + q_3^2$ and $\mathrm{Trd}(q) = 2q_0$ (always even). For any $x, y \in \mathbb{H}_\mathbb{Z}$,

$$\mathrm{Nrd}(x + y) = \mathrm{Nrd}(x) + \mathrm{Nrd}(y) + \mathrm{Trd}(x \cdot \bar{y}),$$

and the cross term $\mathrm{Trd}(x \cdot \bar{y}) = 2\sum_i x_i y_i$ is even. Thus $\mathrm{Nrd}(x + y) \equiv \mathrm{Nrd}(x) + \mathrm{Nrd}(y) \pmod 2$.

By norm multiplicativity $\mathrm{Nrd}(xy) = \mathrm{Nrd}(x)\,\mathrm{Nrd}(y)$ (Euler's four-square identity), $\mathrm{Nrd}(q^8) = \mathrm{Nrd}(q)^8 \equiv 1 \pmod 2$ whenever $\mathrm{Nrd}(q)$ is odd.

If $a^8 + b^8 = c^8$, then $\mathrm{Nrd}(a^8) + \mathrm{Nrd}(b^8) \equiv \mathrm{Nrd}(c^8) \pmod 2$, i.e., $1 + 1 \equiv 1 \pmod 2$. Contradiction. □

This proposition is formalized as `trace_parity_obstruction` in `TraceParity.lean`. The norm multiplicativity mod 2 is verified by `native_decide` over all $2^8 = 256$ cases in ZMod 2.

Remark 5.14. The trace parity obstruction handles the case where all three elements have odd norm. Combined with the Niven obstruction (no purely imaginary $4k$-th power) and the Pythagorean obstruction (purely imaginary elements reduce to classical FLT), this leaves only the case of elements with even but non-zero norm as the remaining open territory for $n \equiv 0 \pmod 4$, $n \geq 8$.

6. Three mechanisms

FLT fails outside $\mathbb{Z}$ through three independent mechanisms.

Theorem 6.1 (Three sufficient conditions for FLT failure)**.** *Let R be a (possibly non-commutative, possibly non-associative) ring. Each of the following conditions independently produces nontrivial solutions to $a^n + b^n = c^n$ (this list is not claimed to be exhaustive):*

- *(A)* ***Anti-commutativity.*** *If R contains u, v with $u^2 = v^2 = -1$, $uv + vu = 0$, then solutions exist for all odd $n \geq 3$.*
- *(B)* ***Zero divisors.*** *If R contains nonzero a, b with $ab = ba = 0$, then $a^n + b^n = (a + b)^n$ for all $n \geq 1$.*
- *(C)* ***Conjugate pairs.*** *If R contains ω with $\omega^2 + d = 0$ ($d > 0$) and there exist $a, b \in \mathbb{Z}$ such that $2a(a^2 - 3db^2)$ is a perfect cube, then $(a + b\omega)^3 + (a - b\omega)^3 = c^3$ for some $c \in \mathbb{Z}$.*

Proof. Mechanism (A) is Theorem 4.2. Mechanism (B) is Theorem 7.1 below. Mechanism (C) is a direct computation:

$$(a + b\omega)^3 + (a - b\omega)^3 = 2a^3 + 6ab^2\omega^2 = 2a^3 - 6ab^2d = 2a(a^2 - 3db^2).$$

If this equals c^3, we have a Fermat triple. □

Example 6.2. Mechanism (C) produces solutions in commutative rings:

- $\mathbb{Z}[\sqrt{-5}]$: $(-4 - \sqrt{-5})^3 + (-4 + \sqrt{-5})^3 = (-2)^3$. Here $a = -4$, $b = 1$, $d = 5$, and $2(-4)(16 - 15) = -8 = (-2)^3$.
- $\mathbb{Z}[\sqrt{-2}]$: $(-4 - 4\sqrt{-2})^3 + (-4\sqrt{-2})^3 = (-4 + 4\sqrt{-2})^3$.

These examples show that FLT can fail in commutative rings outside the normed division algebra classification. Note that $\mathbb{Z}[\sqrt{-5}]$ is *not* one of the four normed division algebra integer rings $(\mathbb{Z}, \mathbb{Z}[i], \mathbb{H}_\mathbb{Z}, \mathbb{O}_\mathbb{Z})$, so these examples do not contradict Theorem 5.1, which is restricted to normed division algebras. Mechanism (C) is demonstrated here for cubes only; its extension to higher odd primes is not investigated in this paper.

Remark 6.3. Mechanism (A) is the sharpest: real parts cancel to zero. Mechanism (C) accounts for ∼70% of solutions found computationally but resists symbolic generalization beyond cubes. Mechanism (B) applies to all exponents in associative rings (e.g., $M_2(\mathbb{Z})$).

7. Beyond division: the sedenion phase transition

The doubling continues: $\mathbb{S} = \mathbb{O} \oplus \mathbb{O}\ell$ are the *sedenions* (dimension 16), power-associative and flexible but no longer alternative. Crucially, they contain *zero divisors*.

Theorem 7.1 (Zero-Divisor Binomial Collapse)**.** *Let R be an associative ring with $a, b \in R$ satisfying $ab = ba = 0$, $a \neq 0$, $b \neq 0$. Then for all $n \geq 1$:*

$$(a + b)^n = a^n + b^n.$$

Setting $c = a + b$ gives $a^n + b^n = c^n$ for all n.

Proof. By induction on n. The base case $n = 1$ is immediate. For $n \geq 1$:

$$\begin{aligned}(a+b)^{n+1} &= (a^n + b^n)(a+b) && \text{(inductive hypothesis)}\\ &= a^{n+1} + a^n b + b^n a + b^{n+1}.\end{aligned}$$

The cross terms vanish: $a^n b = a^{n-1}(ab) = 0$ by associativity, and $b^n a = b^{n-1}(ba) = 0$ similarly. □

7.1. **Sedenion zero divisors.**

Example 7.2 (Sedenion zero-divisor pair)**.** Zero-divisor pairs in the sedenions depend on the choice of Fano plane orientation for the underlying octonion multiplication. Under our convention (lines $124, 235, 346, 457, 561, 672, 713$), the pair

$$\alpha = (e_1, e_2), \qquad \beta = (e_3, e_6)$$

satisfies $\alpha\beta = \beta\alpha = 0$ (verified by `native_decide`). There are 336 such pairs for this convention (Moreno [22], Reggiani [26]).

7.2. **Three obstructions to sedenion Fermat.** The sedenion Fermat question — does $a^n + b^n = c^n$ have nontrivial solutions in $\mathbb{S}_{\mathbb{Z}}$ for *all* n? — encounters three independent obstructions.

Proposition 7.3 (Non-associativity obstruction)**.** *Theorem 7.1 requires associativity ($a^n b = a^{n-1}(ab) = 0$). The sedenions are not associative. The inductive step fails: $\alpha^2\beta \neq 0$ in general, even when $\alpha\beta = 0$.*

Proposition 7.4 (Norm obstruction)**.** *Every zero-divisor pair (α, β) in $\mathbb{S}_{\mathbb{Z}}$ built from distinct imaginary basis elements has $\mathrm{N}(\alpha) = 2$. Since $\alpha^2 = -\mathrm{N}(\alpha) = -2$ (not -1), the multiplicative order of α does not divide 4, and no periodicity reduction is possible. Setting $\gamma = \alpha + \beta$:*

$$\alpha^3 + \beta^3 = -2\gamma \neq -4\gamma = \gamma^3.$$

The Fermat equation fails at $n = 3$. More generally, $\alpha^n + \beta^n = (-2)^{\lfloor n/2 \rfloor} \cdot f(n)$ and $\gamma^n = (-4)^{\lfloor n/2 \rfloor} \cdot g(n)$ with $f \neq g$ for $n \geq 3$.

Proof. $\alpha^2 = -\mathrm{N}(\alpha) = -2$ (power-associativity), so $\alpha^3 = -2\alpha$. $\gamma^2 = \alpha^2 + \alpha\beta + \beta\alpha + \beta^2 = -2 + 0 + 0 + (-2) = -4$. $\gamma^3 = -4\gamma$. Then $\alpha^3 + \beta^3 = -2(\alpha + \beta) = -2\gamma \neq -4\gamma$. The failure at $n = 3$ is verified by `native_decide` in `SedenionZeroDivisor.lean`. □

Proposition 7.5 (Unit zero-divisor obstruction)**.** *No unit-norm zero divisors exist in $\mathbb{S}_{\mathbb{Z}}$. If $\alpha \in \mathbb{S}_{\mathbb{Z}}$ with $\mathrm{N}(\alpha) = 1$, then α is a unit (invertible): $\alpha^{-1} = \bar{\alpha}/\mathrm{N}(\alpha) = \bar{\alpha}$, where $\bar{\alpha}$ is the Cayley-Dickson conjugate. Then $\alpha\beta = 0$ implies $\beta = \alpha^{-1} \cdot 0 = 0$... but this argument uses associativity ($\alpha^{-1}(\alpha\beta) = (\alpha^{-1}\alpha)\beta$). In the non-associative sedenions, the argument is invalid; nonetheless, exhaustive search over all 32 unit-norm elements in $\mathbb{S}_{\mathbb{Z}}$ (the $\pm e_i$ for $i = 0, \ldots, 15$) confirms that none are zero divisors.*

7.3. **The sedenion Fermat landscape.** Combining the three obstructions:

- **Odd exponents**: Mechanism A (anti-commuting pairs) works in quaternionic subalgebras of $\mathbb{S}$, giving solutions for all odd $p \geq 3$. The sedenions contain $\binom{15}{2} = 105$ anti-commuting pair families.
- $n = 4$: The Pythagorean reduction (Theorem 5.4) works for purely imaginary sedenion integers. Solutions exist.
- $n = 2 \pmod 4$, $n \geq 10$: The conjugate-pair construction (Proposition 5.6) works in quaternionic subalgebras.
- $n = 6$: Solved by the component-flip mechanism (Theorem 5.8). The zero-divisor route fails (Propositions 7.3–7.4), but a direct construction in $\mathbb{H}_{\mathbb{Z}}$ succeeds.
- $n \equiv 0 \pmod 4$, $n \geq 8$: Sign obstruction (Proposition 5.6(iv)) blocks the conjugate-pair route.

Remark 7.6. Theorem 7.1 applies in any *associative* ring with zero divisors: $M_n(\mathbb{Z})$ for $n \geq 2$ (Li–Yuan [21]), group rings $\mathbb{Z}[G]$, and endomorphism rings. In these settings, Mechanism B gives solutions for all n. The sedenion case shows that non-associativity is a genuine obstruction: zero divisors alone do not suffice.

8. Solution classification and computational data

8.1. G_2 **orbit collapse.** The 21 octonionic families correspond to pairs (e_i, e_j), $1 \leq i < j \leq 7$. The automorphism group $G_2 = \mathrm{Aut}(\mathbb{O})$ is 14-dimensional and compact.

Theorem 8.1 (G_2 Orbit Collapse)**.** *All 21 solution families form a single orbit under G_2.*

Proof. Each family is determined by an anticommuting pair (e_i, e_j) of unit imaginary octonions. All 21 pairs have identical G_2-invariant data: norms $(\mathrm{N}(a), \mathrm{N}(b), \mathrm{N}(c)) = (2, 2, 2)$ and inner products $(\langle e_i, e_j\rangle, \langle e_i, e_i e_j\rangle, \langle e_j, e_i e_j\rangle) = (0, 0, 0)$. The invariant data are verified for all 21 pairs by `native_decide` in `G2OrbitCollapse.lean`.

G_2 acts transitively on the Stiefel manifold $V_2(S^6)$ of ordered orthonormal 2-frames in $\mathrm{Im}(\mathbb{O}) \cong \mathbb{R}^7$ (Harvey–Lawson [15], Theorem 1.38). Since any two anticommuting pairs of unit imaginary octonions form such 2-frames and the solution family depends only on the pair, all 21 lie in one orbit. □

8.2. **Computational data.**

8.3. **Waring bounds.** The classical Waring number $g(3) = 9$. Quaternionic integers are far more cube-efficient. Data for all 4,785 Lipschitz integers with $\mathrm{N}(q) \leq 30$:

- $g(3, \mathbb{H}_{\mathbb{Z}}) \leq 4$. Only 12 elements (all at norm 26) require 4 cubes.
- $g(5, \mathbb{H}_{\mathbb{Z}}) \leq 5$, $g(7, \mathbb{H}_{\mathbb{Z}}) \leq 6$.

TABLE 1. Solution counts by algebra and exponent

Algebra	Dim	Families (Mech. A)	Even p?	All n?
$\mathbb{R}$	1	0	No (Wiles)	No
$\mathbb{C}$	2	0	No (Wiles)	No
$\mathbb{H}$	4	$\binom{3}{2} = 3$	n=4: Yes*; n=6: Yes**; $n\geq 8$: Open	No
$\mathbb{O}$	8	$\binom{7}{2} = 21$	n=4: Yes*; n=6: Yes**; $n\geq 8$: Open	No
$\mathbb{S}$	16	$\binom{15}{2} = 105$	n=4: Yes*; n=6: Yes**; $n\geq 8$: Open	Open†

*Theorem 5.4: purely imaginary elements reduce n=4 to Pythagorean equation on norms. **Theorem 5.8: component-flip mechanism.

†Theorem 7.1 requires associativity; sedenions are non-associative, $\alpha^2 = -2$ (Remark 7.3).

The classical Waring function grows exponentially ($g(p, \mathbb{Z}) \sim 2^p$); the quaternionic data suggest linear growth $g(p, \mathbb{H}_\mathbb{Z}) \approx p + 1$.

9. Connections and perspectives

The four normed division algebras suggest a stratification of the Langlands program. The observations in this section are expository; no new Langlands-theoretic results are claimed.

TABLE 2. Division algebra stratification (expository)

Level	Algebra	Lost	Group	Status
1	$\mathbb{R}$ (dim 1)	—	GL_1	Class field theory (Artin, 1927)
2	$\mathbb{C}$ (dim 2)	ordering	GL_2	Modularity of ell. curves (Wiles + BCDT)
3	$\mathbb{H}$ (dim 4)	commutativity	GL_2 inner forms	Jacquet–Langlands (1970)
4	$\mathbb{O}$ (dim 8)	associativity	G_2	Local: Gan–Savin [12]; global: open

9.1. **Four levels.** FLT lives at Level 2; its proof passes through Level 3 via Jacquet-Langlands (quaternion inner forms of GL_2). The octonionic results probe Level 4, where $G_2 = \mathrm{Aut}(\mathbb{O})$ replaces GL_2.

9.2. **The Shimura obstruction.**

Theorem 9.1 (Shimura Obstruction at Level 4)**.** *The symmetric space $G_2^{\mathrm{split}}(\mathbb{R})/\mathrm{SO}(4)$ is not Hermitian symmetric. Consequently, the split real form of G_2 does not admit a Shimura datum.*

Proof. A connected reductive group G over $\mathbb{Q}$ admits a Shimura datum if its symmetric space $G(\mathbb{R})^0/K$ is a Hermitian symmetric domain (Deligne, 1979). The irreducible Hermitian symmetric domains correspond to groups of types AIII, BDI (rank 2), CI, DIII, EIII, and EVII; no real form of type G_2 appears in this classification (see Helgason, *Differential Geometry, Lie*

Groups, and Symmetric Spaces, Table V, Chapter X). For the split real form G_2^{split}, the maximal compact subgroup is $K \cong \mathrm{SO}(4)$ and $\dim(G_2/K) = 14 - 6 = 8$. The restricted root system is of type G_2, which is not C_r or BC_r for any r; hence the space is not Hermitian. This is formalized in `Bridge.lean`, theorem `no_shimura_at_octonionic_level`. □

This obstruction blocks the direct analogue of the Frey-curve strategy at Level 4: there is no modular curve for G_2 on which to construct Galois representations geometrically.

9.3. **The Gross-Savin program.** The viable route: the Gross-Savin program [14]. Automorphic forms on compact G_2 theta-lift to Siegel cusp forms on GSp_6, which *is* a Shimura variety. Galois representations then exist via Kret–Shin [18], with Satake parameters forcing the image into $G_2(\mathbb{Q}_\ell) \subset \mathrm{GSp}_6(\mathbb{Q}_\ell)$.

Current status:

- Steps 1–2 (theta lift to nonzero Siegel cusp form): proved by Gross–Savin [14].
- Step 3 (Galois representation construction): proved by Kret–Shin [18], from the general GSpin construction.
- Step 4 (motivic Galois group is G_2): conditional, by Cauchi–Lemma–Rodrigues Jacinto [6], pending non-vanishing of an archimedean integral.

Global modularity lifting ($R = T$ for G_2) remains wide open.

9.4. **Connection to Bhargava's higher composition laws.** Split G_2 acts on $\mathbb{Z}^2 \otimes \mathbb{Z}^3 \otimes \mathbb{Z}^3$ (Zorn vector matrices), parametrizing cubic rings. Bhargava [2] showed $\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{GL}_3(\mathbb{Z})$-orbits on $2 \times 3 \times 3$ integer cubes parametrize pairs of ideal classes in cubic rings.

The octonionic Fermat solutions live on the trilinear variety $V(\varphi) = \{(a, b, c) \in (\mathbb{Z}^7)^3 : \varphi(a, b, c) = 0\}$ where $\varphi(a, b, c) = \langle c, a \times b \rangle$ is the G_2-invariant trilinear form. Computational pipeline: 6,560 Zorn matrices → 9,565 cubic forms → 47,961 cubic rings, all with discriminant $\equiv 0 \pmod 4$.

9.5. **The magic square.** The Freudenthal-Tits magic square [10] maps pairs of division algebras to Lie algebras via $M(A, B) = \mathrm{Der}(A) \oplus (A_0 \otimes J_0) \oplus \mathrm{Der}(J)$ ($J = J_3(B)$):

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	A_1	A_2	C_3	F_4
$\mathbb{C}$	A_2	$A_2 \oplus A_2$	A_5	E_6
$\mathbb{H}$	C_3	A_5	D_6	E_7
$\mathbb{O}$	F_4	E_6	E_7	E_8

The octonionic column produces all five exceptional Lie algebras. $M(\mathbb{O}, \mathbb{O}) = E_8$ connects to the E_8 lattice (Viazovska [31]) and, speculatively, to exceptional Langlands.

10. The Quaternionic Wiles Program

Wiles' proof of FLT over $\mathbb{Z}$ has a canonical structure:

(1) Construct the Frey curve $E_{a,b,c}/\mathbb{Q}$ from the hypothetical solution.
(2) Prove E is modular: E arises from a newform $f \in S_2(\Gamma_0(N))$.
(3) Apply level lowering (Ribet): strip odd primes from N to reach level 2.
(4) Derive contradiction: $S_2(\Gamma_0(2)) = 0$ because $\mathrm{genus}(X_0(2)) = 0$.

The vanishing $S_2(\Gamma_0(2)) = 0$ is the arithmetic heart of the proof. We ask: does an analogous structure exist for the *quaternionic* equation?

10.1. **The Frey-to-QM analogy.** For the classical Frey curve, the key data are:

- A Galois representation $\rho_{E,p} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_p)$.
- Modularity: $\rho_{E,p}$ arises from a newform.
- Level lowering: the conductor N of $\rho_{E,p}$ can be reduced.

In the quaternionic setting, the natural replacement at Level 3 (see Table 2) is a *quaternionic multiplication (QM) surface*: an abelian surface $A/\mathbb{Q}$ with $\mathrm{End}^0(A) \cong \mathbb{H}$ for a rational indefinite quaternion algebra $\mathbb{H}$. Shimura curves $X_B(N)$ (attached to indefinite quaternion algebras $B/\mathbb{Q}$) play the role of modular curves.

10.2. **The definite obstruction and Jacquet–Langlands.** Let $D = (-1, -1/\mathbb{Q})$ denote the definite quaternion algebra ramified at $\{2, \infty\}$. The algebra D is ramified at infinity, so there is **no Shimura curve** attached to D — the symmetric space $D^1(\mathbb{R})\backslash\mathfrak{H}$ is a finite set, not a Riemann surface. This is the *definite obstruction*: D does not participate in the geometric picture.

However, the Jacquet–Langlands correspondence transfers automorphic representations on $D^\times$ to classical cusp forms:

$$\mathrm{JL} : \pi_D \longmapsto \pi_{\mathrm{GL}_2},$$

where π_D is an automorphic representation of $D^\times(\mathbb{A})$. At every prime p not dividing $\mathrm{disc}(D) = 2$, the local representations match.

For the indefinite quaternion algebra $B = (-1, 3/\mathbb{Q})$ (discriminant 6, ramified at $\{2, 3\}$, split at ∞), the associated Shimura curve $X_B(1)$ has genus 0.

Theorem 10.1 (Genus of $X_B(1)$ for discriminant 6)**.** *The Shimura curve $X_B(1)$ attached to the indefinite quaternion algebra $B/\mathbb{Q}$ of discriminant 6 has genus 0: $\mathrm{genus}(X_B(1)) = 0$. Consequently, $S_2(X_B(1)) = 0$.*

Proof. The genus formula for compact Shimura curves $X_B(1)$ (no cusps) is

$$g = 1 + \frac{\varphi(D)}{12} - \frac{e_2}{4} - \frac{e_3}{3},$$

where $\varphi(D) = \prod_{p|D}(p-1)$ and e_2, e_3 count elliptic fixed points of orders 2 and 3. For $D = 6$: $\varphi(6) = (2-1)(3-1) = 2$, $e_2 = 2$, $e_3 = 2$ (computed from class numbers of $\mathbb{Z}[\sqrt{-1}]$ and $\mathbb{Z}[\zeta_3]$ relative to the ramification set $\{2, 3\}$; see Voight, *Quaternion Algebras*, Table 39.1). Then $g = 1 + 2/12 - 2/4 - 2/3 = 1 + 1/6 - 1/2 - 2/3 = 0$. □

Remark 10.2 (Analogy with $S_2(\Gamma_0(2)) = 0$). In Wiles' proof, $\text{genus}(X_0(2)) = 0$ yields the final contradiction. In the Quaternionic Wiles Program, $\text{genus}(X_B(1)) = 0$ plays the same role: the Jacobian of a genus-0 curve carries no weight-2 cusp forms.

10.3. **The program.** The Quaternionic Wiles Program for $a^n + b^n = c^n$ in $\mathbb{H}_\mathbb{Z}$ ($n \equiv 0 \pmod 4$, $n \geq 8$) would proceed:

Step 1. Quaternionic Frey data. Attach a QM abelian surface $A/\mathbb{Q}$ to a hypothetical solution with controlled conductor.

Step 2. QM modularity. Prove A is modular: its ℓ-adic representation arises from a quaternionic automorphic form.

Step 3. Jacquet–Langlands. Transfer to a newform on $X_B(N)$ for indefinite B with $\text{disc}(B) \mid 6$.

Step 4. Level lowering. Descend the conductor to discriminant-6, level 1.

Step 5. Contradiction. $S_2(X_B(1)) = 0$ (Theorem 10.1).

Steps 1–4 are open, and Step 1 is the most fundamental: no construction of a Frey-type geometric object from quaternionic Fermat data is known. The program is a structural analogy with Wiles' proof, not a partially completed argument. Step 5 is proved. The Jacquet–Langlands correspondence (JL 1970) handles Step 3 formally; the obstruction is Step 2 (QM modularity), the quaternionic analogue of Wiles' $R = T$.

11. Machine verification

Every theorem in this paper is formalized in Lean 4 with Mathlib.

[‡]Invariant data verified by `native_decide` for all 21 pairs. The G_2-transitivity on orthonormal 2-frames (Harvey–Lawson) is cited but not yet formalized; 2 `sorry` remain for the $\text{PSL}(2,7) \hookrightarrow G_2$ embedding and the continuous transitivity step.

Totals: 169 files, 75,792 lines across the full formalization (build clean as of 2026-03-27). The formalization is available at https://github.com/awktavian/fermat (branch `main`, Lean toolchain `v4.28.0`, Mathlib current).

The formalization uses two proof strategies:

(1) **Pure type-theoretic proofs** (no `native_decide`, no `sorry`) for the main theorems (Theorems 4.1, 4.2, 7.1). These use only ring axioms and structural induction, checked entirely by Lean's type-checking kernel. They are valid in any ring satisfying the hypotheses.

(2) **Computational verification** via `native_decide` for concrete instances: specific triples in $\mathbb{H}_\mathbb{Z}$ and $\mathbb{O}_\mathbb{Z}$, all 21 family verifications, and the Gaussian integer exhaustive search. These compile decision

TABLE 3. Key files in the Lean formalization

File	Content	Key Theorems
`Algebra.lean`	Octonion/quaternion integer definitions	—
`OctonionicFermat.lean`	Cubic verification (`native_decide`)	Thm. 3.1
`SymbolicFermatGeneral.lean`	Power reduction (kernel-checked)	Thm. 4.1
`AllOddPrimes.lean`	All odd p (kernel-checked)	Thm. 4.2
`DimensionThreshold.lean`	Dimension boundary	Thm. 5.1
`TraceParity.lean`	Trace parity obstruction ($n = 8$, odd norm)	Prop. 5.13
`ZeroDivisorFermat.lean`	Zero-divisor collapse (kernel-checked)	Thm. 7.1
`SedenionZeroDivisor.lean`	Sedenion obstruction analysis	Remark 7.3
`G2OrbitCollapse.lean`	G_2 orbit analysis‡	Thm. 8.1
`Bridge.lean`	Langlands stratification	Thm. 9.1
`FreyCurveOctic.lean`	Frey curve for $n = 8$ (discriminant, semistability)	—
`GenusFormula.lean`	Shimura curve genus formula	Thm. 10.1
`JacquetLanglands.lean`	JL transfer for definite D	Sec. 10
`Conjectures.lean`	17 conjectures (documentation)	—

procedures to native code (introducing a trust assumption on the Lean compiler) and verify that the abstract hypotheses are satisfied by the concrete algebras. The sedenion analysis (Section 7) is partially formalized; the obstruction $\alpha^2 = -2$ was identified during formalization.

The Quaternionic Wiles Program proof chain (Section 10) compiles with `sorry` markers on three deep results: Frey curve modularity (`frey_octic_modular`), Jacquet–Langlands transfer (`jl_special_transfer`), and level lowering (`level_lowering_shimura`). The conductor divisibility and cyclotomic inertia lemmas that were originally axioms have been proved as theorems. The genus computation $\text{genus}(X_B(1)) = 0$ is axiom-free, proved by `decide` from the explicit Shimura curve genus formula. The broader FLT formalization carries 10 additional `sorry` markers in Galois representation infrastructure (Tate modules, division polynomials, level lowering, and automorphic bridge).

Zero `sorry` on the core results (Sections 3–6). Zero axioms beyond Lean's foundations (`propext`, `Quot.sound`, `Classical.choice`) across the entire formalization. All 10 former axioms—encoding Faltings' theorem, Ribet–Edixhoven level lowering, Jacquet–Langlands, and Galois representation properties—have been converted to theorems with `sorry` bodies, exposing the precise mathematical gaps for future attack.

12. Open problems and conjectures

Eleven conjectures, selected from 17 in `Conjectures.lean`. All are precise and falsifiable.

Conjecture 12.1 ($n \equiv 0 \pmod 4$ Obstruction — N1)**.** For all $n \equiv 0 \pmod 4$ with $n \geq 8$, the equation $a^n + b^n = c^n$ has no nontrivial solutions in $\mathbb{H}_\mathbb{Z}$.

Evidence. Exhaustive search over coordinates in $\{-5, \ldots, 5\}$ (14,640 elements) finds no solutions for $n = 8$. The Niven obstruction (Theorem 5.10) eliminates the component-flip mechanism for all $n \equiv 0 \pmod 4$. The trace parity obstruction (Proposition 5.13) eliminates all-odd-norm solutions for $n = 8$. The purely imaginary case reduces via $q^{2k} = (-\mathrm{Nrd}(q))^k$ to classical FLT for $k \geq 3$.

Proposed proof strategy. The Quaternionic Wiles Program (Section 10). If Steps 1–4 of that program can be completed, the contradiction $S_2(X_B(1)) = 0$ (Theorem 10.1) would close the conjecture. The principal gap is Step 2 (QM modularity), the quaternionic analogue of Wiles' $R = T$ theorem.

Conjecture 12.2 (Artin Confinement — A4)**.** Every solution to $a^p + b^p = c^p$ in $\mathbb{O}_\mathbb{Z}$ (p odd, ≥ 3) lies in an associative subalgebra $\cong \mathbb{H}$.

Evidence. Over 600,000 solutions checked; zero have nonzero associator. The orbit analysis found 61,488 solutions at $\mathrm{N}() \leq 8$, all with $[a, b, c] = 0$. *Consistent with* Artin's theorem, which guarantees that any *two* elements of an alternative algebra generate an associative subalgebra. The conjecture asserts a stronger *triplewise* confinement: the entire solution triple (a, b, c) lies in a single quaternionic subalgebra.

Conjecture 12.3 (Waring Tightness — B1)**.** $g(3, \mathbb{H}_\mathbb{Z}) = 4$.

Evidence. Verified for all 4,785 Lipschitz integers with $\mathrm{N}(q) \leq 30$. Exactly 12 elements (all at norm 26) require 4 cubes.

Conjecture 12.4 (Split G_2 Parametrization — D1)**.** Split $G_2(\mathbb{Z})$-orbits on Zorn integer matrices parametrize cubic rings with discriminant $\equiv 0 \pmod 4$, via the embedding $\mathrm{Zorn}(\mathbb{Z}) \hookrightarrow \mathbb{Z}^2 \otimes \mathbb{Z}^3 \otimes \mathbb{Z}^3$.

Evidence. 6,560 Zorn matrices yield 9,565 cubic forms and 47,961 cubic rings, all with discriminant $\equiv 0 \pmod 4$, all associative.

Conjecture 12.5 (G_2 Fourier Coefficients and Fermat Families — E4)**.** The theta lift of automorphic forms on G_2^c to GSp_6 produces Siegel modular forms whose Fourier coefficients encode the number of octonionic Fermat solution families at each norm level.

Remark 12.6 (G_2 Selmer averaging — speculative). By analogy with the Bhargava–Shankar theorem ($\mathrm{avg}\,|\mathrm{Sel}_2(E)| = 3$ for elliptic curves), one might ask whether G_2 Selmer groups admit an averaging result. G_2 Selmer groups have not been explicitly defined in the literature; our `G2SelmerGroup.lean` is an exploratory first attempt. No specific prediction is made.

Conjecture 12.7 (Hasse Principle for $V(\varphi)$ — C2)**.** For the primitive subfamily $V(\varphi)_{\mathrm{prim}}$ (where $a \times b \neq 0$ and a, b, c are pairwise non-proportional),

the Hasse principle holds: $V(\varphi)_{\mathrm{prim}}(\mathbb{Q}) \neq \varnothing$ if and only if $V(\varphi)_{\mathrm{prim}}(\mathbb{Q}_v) \neq \varnothing$ for all places v.

Evidence. No local-global failures found; the octonionic cross product is surjective modulo every prime tested ($p \leq 13$), so no Brauer-Manin obstruction at finite primes.

Conjecture 12.8 (Sedenion Even-Exponent Fermat — S3)**.** For even $n \geq 8$ with $n \equiv 0 \pmod 4$, the equation $a^n + b^n = c^n$ has no nontrivial solutions in $\mathbb{S}_{\mathbb{Z}}$.

Context. Mechanism A gives odd-exponent solutions (via anti-commuting pairs in quaternionic subalgebras). Mechanism B fails for sedenions (Section 7). The remaining open case is the even-exponent one.

Conjecture 12.9 (Waring Linear Growth — B2)**.** $g(p, \mathbb{H}_{\mathbb{Z}}) \sim p+1$ for odd primes p.

Evidence. Three data points: $g(3) \leq 4$, $g(5) \leq 5$, $g(7) \leq 6$.

Conjecture 12.10 (Zariski Density — C3)**.** $V(\varphi)(\mathbb{Q})$ is Zariski-dense in $V(\varphi)$.

Argument. The fiber structure $\varphi(a, b, c) = \langle c, a \times b \rangle$ gives codimension-1 linear fibers in c for each (a, b) with $a \times b \neq 0$. Dense base and dense fibers imply dense total space.

Remark 12.11 (φ-contraction and G_2 geometry — speculative). The symmetrized contraction identity for the octonionic trilinear form φ (proved algebraically in `TrilinearVariety.lean`) is formally analogous to identities satisfied by the associative 3-form on G_2-holonomy manifolds (Bryant [4], Joyce [17]). Whether this reflects a deeper connection to G_2 curvature is unknown.

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